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Spectral modelling of homogeneous non-isotropic turbulence

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The paper describes a method to calculate homogeneous anisotropic turbulent fields associated with a constant mean velocity gradient. The equations governing the Fourier transform of the triple velocity correlations are closed by using an extended eddy-damped quasi-normal approximation. An angular parametrization of the second-order spectral tensor is introduced in order to integrate analytically all the directional terms over a spherical shell. Numerical solutions of the model are presented for typical homogeneous anisotropic flows.

1. Introduction

The problem of closure in spectral space has been considered in several ways. The quasi-normal hypothesis associated with a damping term (EDQNM) is one of the most straightforward treatments. Originally the quasi-normal hypothesis was introduced by Millionschikov (1941), then by Proudman & Reid (1954) for the case of isotropic turbulence. O'Brien & Francis (1962) and Ogura (1963) brought to light some inconsistencies: the assumption overestimates the transfer term which generates a negative part in the turbulent energy spectrum. This first theory was improved by Orszag (1970) who introduced a damping term, which moderates the growth of the spectral transfer Later most refinements were made by several authors (Leith 1971; André & Lesieur 1977), and striking results were thus obtained for the case of an isotropic turbulent field. Our purpose is to extend this sort of assumption to the case of a homogeneous anisotropic turbulence associated with a constant mean velocity gradient.

This approach, which starts from the equations of velocity correlations at several points and their Fourier transforms (Burgers & Mitchner 1953), is supported by the exhaustive analytical investigation of Craya (1958).

The equations governing the triple velocity correlations are presented. They are closed by using an extended eddy-damped quasi-normal approximation. Even in the case of non-isotropic fields, the relaxation of the triple correlations is only taken into account through a single eddy-damping coefficient.

These equations are cast into a tractable form in order to carry out a computing treatment. The turbulent structures are be considered according to their sizes only. In fact, a parametrization of the second-order spectral tensor is introduced which displays wavenumber directivity. Consequently, integration over a sphere of radius K is performed analytically. At this stage a new scalar parameter appears.

The adjustments of the two previous parameters are made by reference to the decay of isotropic turbulence and to the rapid distortion of an initially isotropic turbulence.

The method is applied to several cases, some interesting comparisons with experimental data are chosen. The results of Gence (1979) are considered; the sudden change in the direction of the principal axes of the straining process is analysed. The way in which an anisotropic turbulence turns into an isotropic state is also examined in order to clarify the role played by nonlinear terms.

2. The eddy-damped quasi-normal approximation for homogeneous non-isotropic turbulence

The flow under consideration is incompressible; moreover the turbulent field is supposed to be homogeneous. All the variables can be split into two parts which respectively characterize the mean and the fluctuating fields:

$$V_i = \langle V_i \rangle + v_i.$$

The mean velocity is given by

$$\langle V_i(\mathbf{x},t)\rangle = V_i^0 + \lambda_{il} x_l;$$

V_i^0 and the mean velocity gradient λ_{il} are independent of $\mathbf{x}$ and t and the Fourier transform of the Navier-Stokes equation for the fluctuation u_i can be written as

$$\left(\frac{\partial}{\partial t} + \nu K^2\right)\hat{v}_i + \lambda_{il}\hat{v}_l + iK_l V_l^0 \hat{v}_i - 2\frac{K_i K_m}{K^2}\lambda_{ln}\hat{v}_l - \lambda_{ln}K_l \frac{\partial \hat{v}_i}{\partial K_n}$$
$$= -iK_l\left(\delta_{in} - \frac{K_i K_n}{K^2}\right)\int_{\mathbb{R}^6} \hat{v}_n(\mathbf{P})\,\hat{v}_l(\mathbf{Q})\,\delta(\mathbf{P}+\mathbf{Q}-\mathbf{K})\,d^3\mathbf{P}\,d^3\mathbf{Q}, \quad (2.1)$$

wherein ν is the molecular viscosity, $\hat{v}_i(\mathbf{K},t)$ is the Fourier transform of the three-dimensional velocity field

$$\hat{v}_i(\mathbf{K},t) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} v_i(\mathbf{x},t)\exp(-i\mathbf{K}.\mathbf{x})\,d^3\mathbf{x}.$$

The two terms, on the right-hand side of equation (2.1), respectively represent the turbulent advective effects and the nonlinear part of the pressure mechanisms.

We now introduce the Fourier transform of the correlation functions

$$\phi_{ij}(\mathbf{K},t) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \langle v_i(\mathbf{x},t)\,v_j(\mathbf{x}+\mathbf{r},t)\rangle \exp(-i\mathbf{K}.\mathbf{r})\,d^3\mathbf{r},$$

$$\phi_{ijl}(\mathbf{K},\mathbf{P},t) = \frac{i}{(2\pi)^6}\int_{\mathbb{R}^6} \langle v_i(\mathbf{x},t)\,v_j(\mathbf{x}+\mathbf{r},t)\,v_l(\mathbf{x}+\mathbf{r}',t)\exp[-i(\mathbf{K}.\mathbf{r}+\mathbf{P}.\mathbf{r}')]\,d^3\mathbf{r}\,d^3\mathbf{r}',$$

connected with $\hat{v}_i(\mathbf{K})$ through the relations

$$\left.\begin{aligned}\langle \hat{v}_i(\mathbf{P},t)\,\hat{v}_j(\mathbf{K},t)\rangle &= \delta(\mathbf{K}+\mathbf{P})\,\phi_{ij}(\mathbf{K},t),\\ i\langle \hat{v}_i(\mathbf{Q},t)\,\hat{v}_j(\mathbf{K},t)\,\hat{v}_l(\mathbf{P},t)\rangle &= \delta(\mathbf{K}+\mathbf{P}+\mathbf{Q})\,\phi_{ijl}(\mathbf{K},\mathbf{P},t).\end{aligned}\right\} \quad (2.2)$$

Multiplying (2.1) by suitable products of Fourier modes, we obtain, by taking ensemble averages and introducing previous definitions (2.2),

$$\left(\frac{\partial}{\partial t} + 2\nu K^2\right)\phi_{ij} + \psi_{ij} = \Omega_{ij}, \quad (2.3)$$

$$\left[\frac{\partial}{\partial t}+\nu(K^2+P^2+Q^2)\right]\phi_{ijl}+\psi_{ijl} = \Omega_{ijl}, \tag{2.4}$$

where

$$\psi_{ij} = \lambda_{il}\phi_{lj}+\lambda_{jl}\phi_{il}-\lambda_{ln}K_l\frac{\partial}{\partial K_n}\phi_{ij}-2\lambda_{ln}\frac{K_l}{K^2}(K_i\phi_{nj}+K_j\phi_{in}),$$

$$\Omega_{ij} = t_{ij}+t^*_{ji}-\frac{K_n}{K^2}(K_it_{nj}+K_jt^*_{ni}),$$

with

$$t_{ij}(\mathbf{K},t) = iK_l\int_{\mathbb{R}^3}\phi_{jli}(\mathbf{K},\mathbf{P},t)\,d^3\mathbf{P}, \tag{2.5}$$

and

$$\begin{aligned}\psi_{ijl} = {} & \lambda_{in}\phi_{njl}+\lambda_{jn}\phi_{inl}+\lambda_{ln}\phi_{ijn}\\ & +2\lambda_{nm}\left(\frac{Q_iQ_n}{Q^2}\phi_{mjl}+\frac{K_jK_n}{K^2}\phi_{iml}+\frac{P_lP_n}{P^2}\phi_{ijm}\right)\\ & -\lambda_{nm}\left[K_n\frac{\partial}{\partial K_m}\phi_{ijl}+P_n\frac{\partial}{\partial P_m}\phi_{ijl}\right],\end{aligned}$$

$$\Omega_{ijl} = Q_n\left(\delta_{im}-\frac{Q_iQ_m}{Q^2}\right)\theta_{nmjl}+K_n\left(\delta_{jm}-\frac{K_jK_m}{K^2}\right)\theta'_{nmli}+P_n\left(\delta_{lm}-\frac{P_lP_m}{P^2}\right)\theta''_{nmij},$$

with the definitions

$$\theta_{lijk}(\mathbf{K},\mathbf{P},t) = \frac{1}{(2\pi)^6}\int_{\mathbb{R}^6}\langle v_l(\mathbf{x})\,v_i(\mathbf{x})\,v_j(\mathbf{x}+\mathbf{r})\,v_k(\mathbf{x}+\mathbf{r}')\rangle\exp\left[-i(\mathbf{K}.\mathbf{r}+\mathbf{P}.\mathbf{r}')\,d^3\mathbf{r}\,d^3\mathbf{r}';\right.$$

$$\theta'_{lijk} = \theta_{lijk}(\mathbf{P},\mathbf{Q},t);\quad \theta''_{lijk} = \theta_{lijk}(\mathbf{Q},\mathbf{K},t);\quad \mathbf{K}+\mathbf{P}+\mathbf{Q} = 0.$$

The problem is closed if the fourth-order correlations are expressed in terms of third- and second-order correlations; assuming that the joint probability distribution at three points is not too far from a normal law, the fourth-order cumulant is supposed to be linearly dependent on the third-order cumulant (Sulem, Frisch & Lesieur 1975). We have then in spectral space:

$$\begin{aligned}C^4_{nmjl} = {} & \theta_{nmjl}(\mathbf{K},\mathbf{P},t)-\langle v_nv_m\rangle\,\delta(\mathbf{K}+\mathbf{P})\,\phi_{jl}(-\mathbf{K},t)-\phi_{nj}(\mathbf{K},t)\,\phi_{ml}(\mathbf{P},t)\\ & -\phi_{nl}(\mathbf{P},t)\,\phi_{mj}(\mathbf{K},t) = \mathscr{L}_{nmjlpqr}\phi_{pqr}(\mathbf{K},\mathbf{P},t).\end{aligned}$$

With the above hypothesis, equations (2.4) can be written

$$\left[\frac{\partial}{\partial t}+\nu(K^2+P^2+Q^2)\right]\phi_{ijl}+\psi_{ijl} = \Omega^{QN}_{ijl}-L^{ED}_{ijlpqr}\phi_{pqr}, \tag{2.6}$$

with

$$\Omega^{QN}_{ijl} = Q_n\left(\delta_{im}-\frac{Q_iQ_m}{Q^2}\right)\{\phi_{nj}(K,t)\,\phi_{ml}(P,t)+\phi_{nl}(P,t)\,\phi_{mj}(K,t)\}+\dots,$$

$$L^{ED}_{ijlpqr} = -Q_n\left(\delta_{im}-\frac{Q_iQ_m}{Q^2}\right)\mathscr{L}_{nmjlpqr}(\mathbf{K},\mathbf{P},t)+\dots.$$

In the right-hand side of equation (2.6), the first term incorporates double correlations only, whereas the second term takes into account the departure from a normal law through a linear relaxation.

The three terms $\nu(K^2+P^2+Q^2)\,\phi_{ijl}$, ψ_{ijl} and $L^{ED}_{ijlpqr}\phi_{pqr}$ are written in the compact form

$$\nu(K^2+P^2+Q^2)\,\phi_{ijl}+\psi_{ijl}+L^{ED}_{ijlpqr}\phi_{pqr} = \mu(K,P,Q,t)\,\phi_{ijl}, \tag{2.7}$$

where μ introduces a scalar eddy damping rate similar to that initially proposed by Orszag (1970)

$$\mu(K,P,Q,t) = \nu(K^2+P^2+Q^2)+\eta(K,t)+\eta(P,t)+\eta(Q,t);$$

therefore we find that

$$\left(\frac{\partial}{\partial t}+\mu\right)\phi_{ijl} = \Omega^{QN}_{ijl}. \tag{2.8}$$

The model may then be regarded as a simplified form of the generalized test field model (Kraichnan 1972) in which different damping factors are determined from a dynamical equation. In the previous assumption (2.7) the incorporation of the typically non-isotropic term ψ_{ijl} in the diagonalized form has to be discussed especially. We can remark that the mean velocity gradient is taken into account in the equation governing the double correlations. This effect is thus previously introduced in the equation (2.8) through the term Ω^{QN}_{ijl}. Then the variation of ϕ_{ijl} with time should be primarily controlled by Ω^{QN}_{ijl}. Moreover in equation (2.3), we are only concerned with a part (relation 2.5) of the triple velocity correlations weighted by the wavenumber, and thus the behaviour of equation (2.4) has to be considered especially for the case of large and moderate wavenumbers. Accordingly the assumption

$$\nu(K^2+P^2+Q^2)\,\phi_{ijl}+L^{ED}_{ijlpqr}\phi_{pqr} \gg \psi_{ijl}$$

should be verified at least in this range.

In any case, this closure process can be considered as an extension of the eddy damped quasi-normal theory to turbulent fields subjected to small velocity gradients. The validity of this approach will later be verified by comparisons with experimental data.

For the solution of equation (2.8) it is found that

$$\phi_{ijl}(\mathbf{K},\mathbf{P},t) = \phi_{ijl}(\mathbf{K},\mathbf{P},0)\exp\left(-\int_0^t \mu\,dt'\right) + \int_0^\tau \exp\left(\int_0^\tau \mu\,dt' - \int_0^t \mu\,dt'\right)\Omega^{QN}_{ijl}(\mathbf{K},\mathbf{P},\tau)\,d\tau.$$

From the definition (2.5) of t_{ij} it follows that

$$t_{ij}(\mathbf{K},t) = \int_{\mathbb{R}^3} K_l\phi_{jli}(\mathbf{K},\mathbf{P},0)\exp\left(-\int_0^t \mu\,dt'\right)d^3\mathbf{P} + \int_0^t d\tau\int_{\mathbb{R}^3}\exp\left(\int_0^\tau \mu\,dt' - \int_0^t \mu\,dt'\right)K_l\,\Omega^{QN}_{jli}(\mathbf{K},\mathbf{P},\tau)\,d^3\mathbf{P}.$$

For large values of t the Markovian assumption yields

$$t_{ij}(\mathbf{K},t) = K_l\int_{\mathbb{R}^3}\theta_{KPQ}\,\Omega^{QN}_{jli}(\mathbf{K},\mathbf{P},t)\,d^3P \quad \text{when} \quad \theta_{KPQ} = \mu^{-1},$$

and the equation (2.3) is closed if the eddy damping rate η is known.

3. Parametrization of angular dependence of ϕ_{ij} with K

In order to limit the computing time, an essential simplification in applying the theory is to integrate analytically the closed equation (2.3) over a sphere of radius K.

For the new averaged function,

$$\varphi_{ij}(K,t) = \iint_{S_k} \phi_{ij}(\mathbf{K},t)\, dA(\mathbf{K}),$$

we have

$$\left(\frac{\partial}{\partial t} + 2\nu K^2\right)\varphi_{ij} + \lambda_{il}\varphi_{lj} + \lambda_{jl}\varphi_{il} = P_{ij}^{(1)} + P_{ij}^{(2)} + S_{ij}^{(1)} + S_{ij}^{(2)}, \tag{3.1}$$

with

$$\left.\begin{aligned} P_{ij}^{(1)}(K,t) &= 2\lambda_{ln}\iint_{S_k} \frac{K_l}{K^2}[K_i\phi_{nj}(\mathbf{K},t) + K_j\phi_{in}(\mathbf{K},t)]\, dA(\mathbf{K}),\\ S_{ij}^{(1)}(K,t) &= \lambda_{ln}\iint_{S_k} K_l\frac{\partial}{\partial K_n}[\phi_{ij}(\mathbf{K},t)]\, dA(\mathbf{K}),\\ P_{ij}^{(2)}(K,t) &= \iint_{S_k}\frac{K_lK_n}{K^2}\int_{\mathbb{R}^3}\theta_{KPQ}[K_i\Omega_{ljn}^{QN}(\mathbf{K},\mathbf{P},t) + K_j\Omega_{lin}^{QN}(\mathbf{K},\mathbf{P},t)]\, d^3\mathbf{P}\, dA(\mathbf{K}),\\ S_{ij}^{(2)}(K,t) &= \iint_{S_k} K_l\int_{\mathbb{R}^3}\theta_{KPQ}[\Omega_{jli}^{QN}(\mathbf{K},\mathbf{P},t) + \Omega_{ilj}^{QN}(\mathbf{K},\mathbf{P},t)]\, d^3\mathbf{P}\, dA(\mathbf{K}). \end{aligned}\right\} \tag{3.2}$$

In order to perform an analytical integration of the right-hand side of (3.1) the representation theorems are used for ϕ_{ij}. At first, we have to make a convenient choice of the arguments to be introduced. We take the following representation:

$$\phi_{ij}(\mathbf{K},t) = \frac{E(K,t)}{4\pi K^2}\mathscr{T}_{ij}[H_{pq}(K,t), \alpha_n, C_{lm}(t)]. \tag{3.3}$$

The kinetic-energy spectrum E is used as the single-dimensional scalar function for ϕ_{ij}. The non-dimensional function $\mathscr{T}_{ij}$ can be considered as an extended form of the exact representation of ϕ_{ij} for isotropic turbulence. $\mathscr{T}_{ij}$ depends on angular arguments $\alpha_m = K_m/K$ of the vector $\mathbf{K}$. The dimensionless deviatoric tensor $\mathbf{H}$

$$H_{ij}(K,t) = \frac{\phi_{ij}(K,t)}{2E(K,t)} - \frac{\delta_{ij}}{3}; \quad E(K,t) = \frac{\varphi_{ll}(K,t)}{2},$$

takes explicitly into account anisotropic effects. The second-order tensor $\mathbf{C}$ is deduced from an analysis of the formal behaviour of the solutions of the linear equation (2.3) (obtained when the nonlinear term t_{ij} is dropped out). $\mathbf{C}$ is found to be

$$C_{ij}(t) = F_{li}^{-1}(t)\, F_{lj}^{-1}(t);$$

$\mathbf{F}$ depends on the mean field through the equation

$$\frac{\partial F_{ij}}{\partial t} = \lambda_{il}F_{lj}$$

and introduces a time memory.

These three arguments are considered as being sufficiently representative of the turbulent field as to admit $\mathscr{T}_{ij}$ to be an isotropic tensorial function. Rigorously, ϕ_{ij} includes both a real symmetrical even part and an imaginary antisymmetrical odd one (zero in the case of real initial data). The imaginary contribution appears only in the expressions of $P^{(2)}_{ij}$ and $S^{(2)}_{ij}$ under a quadratic form. Retaining then a first-order expansion with H_{ij} for $\mathscr{T}_{ij}$ and for the final expression of the terms (3.2), it is consistent to neglect the imaginary part; accordingly we use the tensorial expression (Cambon 1979)

$$\mathscr{T}_{ij} = (\delta_{il}-\alpha_i\alpha_l)(\delta_{jn}-\alpha_j\alpha_n)[\delta_{ln}\{1-(15+7a)H_{pq}\alpha_p\alpha_q\}-2aH_{ln}], \tag{3.4}$$

retaining the following formal properties:

$$\alpha_i\mathscr{T}_{ij} = 0; \quad \alpha_j\mathscr{T}_{ij} = 0$$

and

$$\iint_{S_k} \mathscr{T}_{ij}\, dA(\mathbf{K}) = 8\pi K^2\left(\frac{\delta_{ij}}{3}+H_{ij}\right).$$

The scalar parameter a must be expressed in terms of the invariants relative to **H** and **C**.

4. Closed equation for the second-order tensor φ_{ij}

With a defined form for ϕ_{ij} the right-hand side of the basic equation (3.1) can be calculated analytically. $S^{(1)}_{ij}$ and $P^{(1)}_{ij}$ are written in terms of general integrals, the properties of which are underlined:

$$\iint_{S_k} T^{(k,t)}_{iji_1i_2\ldots,i_{2n}}\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{2n}} = T^{(k,t)}_{iji_1i_2\ldots i_{2n}}\iint_{S_k}\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{2n}}\,dA(\mathbf{K})$$

with $n \leqslant 3$ and $i_z \in [1,2,3]$
and

$$\iint_{S_k}\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{2n}}\,dA(\mathbf{K}) = \frac{4\pi K^2}{1.3\ldots(2n-1)}\delta^n_{i_1i_2\ldots i_{2n}},$$

where $\boldsymbol{\delta}^n$ is a sum of Kronecker symbols products defined by

$$\delta^1_{ij} = \delta_{ij},$$

$$\delta^n_{i_1i_2\ldots i_{2n}} = \sum_{\alpha=1}^{2n-1}\delta_{i_\alpha i_{2n}}\delta^{n-1}_{i_1i_2\ldots i_{\alpha-1}i_{\alpha+1}\ldots i_{2n-1}}.$$

Accordingly, we have

$$\left.\begin{aligned} P^{(1)}_{ij} &= 2E\left[\tfrac{2}{5}d_{ij}-3D\{d_{il}H_{lj}+d_{jl}H_{li}-\tfrac{2}{3}\delta_{ij}d_{lm}H_{lm}\}+\tfrac{14}{3}(D+\tfrac{4}{7})(\omega_{il}H_{lj}+\omega_{jl}H_{li})\right],\\ S^{(1)}_{ij} &= -\tfrac{2}{15}d_{ij}\frac{\partial}{\partial K}(KE)+2d_{il}\frac{\partial}{\partial K}(KDEH_{lj})+2d_{jl}\frac{\partial}{\partial K}(KDEH_{li})\\ &\qquad -\frac{\delta_{ij}}{3}d_{lm}\frac{\partial}{\partial K}[(2+11D)KEH_{lm}],\end{aligned}\right\} \tag{4.1}$$

with

$$d_{ij} = \tfrac{1}{2}(\lambda_{ij}+\lambda_{ji}); \quad \omega_{ij} = \tfrac{1}{2}(\lambda_{ij}-\lambda_{ji}); \quad D(K,t) = \tfrac{2}{7}[1+\tfrac{4}{5}a(K,t)].$$

The formal expression of $P^{(1)}_{ij}$ given by (4.1) is similar to that proposed by Lumley (1975) and Launder, Reece & Rodi (1975) for the velocity pressure correlation term in physical space.

In (3.2) the integration $\int_{\mathbb{R}^3} d^3\mathbf{P}$ is simplified by using the new variables

$$\mathbf{K}, P_1, P_2, P_3 \rightarrow \mathbf{K}, P, Q, \theta.$$

For a fixed orientation of **K**, the angular position of the plane, in which is located the triadic system K, P, Q, is defined by θ.

We introduce

$$\alpha_i = K_i/K; \quad \beta_i = P_i/P; \quad \gamma_i = Q_i/Q; \quad x = -\beta_l\gamma_l; \quad y = -\alpha_l\gamma_l; \quad z = -\alpha_l\beta_l,$$

and the expressions for $P_{ij}^{(2)}$ and $S_{ij}^{(2)}$ reduce to

$$T_{ij}(K,t) = \iint_{S_k} dA(\mathbf{K}) \int_{\mathbb{R}^3} \tau_{iji_1 i_2 \ldots i_p j_1 j_2 \ldots j_q}^{(K,P,Q,t)} \alpha_{i_1} \ldots \alpha_{i_p} \beta_{j_1} \ldots \beta_{j_q} d^3\mathbf{P},$$

so that it follows that

$$T_{ij}(K,t) = \iint_{\Delta_k} dP\,dQ \frac{PQ}{K} \tau_{iji_1\ldots i_p j_1 \ldots j_q}^{(K,P,Q,t)} \iint_{S_k} dA(\mathbf{K})\, \alpha_{i_1} \ldots \alpha_{i_p} \int_0^{2\pi} \beta_{j_1} \ldots \beta_{j_q} d\theta.$$

The integration in terms of P and Q has to be performed over the area Δ_k, so that **K**, **P**, **Q** form a triangle.

The integral $\int_0^{2\pi} d\theta$ of products of unit vector components β_i is expressed in terms of K, P, Q and products of components α_i. Consequently the integrals $\iint_{S_k} dA(\mathbf{K})$ are calculated by the above-mentioned technique.

A first-order expansion with respect to **H** gives

$$P_{ij}^{(2)}(K,t) = 2 \iint_{\Delta_k} \theta_{KPQ}\, p_{ij}(K,P,Q,t) \frac{dP\,dQ}{PQ},$$

$$S_{ij}^{(2)}(K,t) = 2 \iint_{\Delta_k} \theta_{KPQ}\, S_{ij}(K,P,Q,t) \frac{dP\,dQ}{PQ},$$

with

$$p_{ij} = (x+yz)\, H''_{ij} \left[K^2PE'E'' \left\{ y(z^2-y^2)(a''+3) + (y+xz)\frac{a''}{5} \right\} - P^3EE''y(z^2-x^2)(a''+3) \right],$$

$$S_{ij} = (xy+z^3) \left\{ K^2PE'E'' \left(\frac{\delta_{ij}}{3} + H'_{ij} + H''_{ij} \right) - P^3EE'' \left(\frac{\delta_{ij}}{3} + H_{ij} + H''_{ij} \right) \right\}$$
$$+ H''_{ij}(K^2PE'E''C_{KPQ} - P^3EE''C_{PKQ})$$

$$C_{KPQ} = (xy+z)\{(y^2-z^2)(a''+3) + \tfrac{2}{5}a''(1+z^2)\}; \quad E' = E(P,t); \quad E'' = E(Q,t); \ldots.$$

5. The η and a functions

These two unkown functions $\eta(K,t)$ and $a(K,t)$ are determined by referring to two limiting cases: the decay of an isotropic turbulence, and the behaviour of a turbulent field subject to a rapid distortion.

For the case of isotropic turbulence λ_{ij}, H_{ij}, $P_{ij}^{(1)}$, $P_{ij}^{(2)}$ and $S_{ij}^{(1)}$ are zero.

The spectrum tensor ϕ_{ij} can be written

$$\phi_{ij}(\mathbf{K},t) = \frac{E(K,t)}{4\pi K^2}(\delta_{ij} - \alpha_i\alpha_j).$$

It is defined by the single scalar function $E(K,t)$. Therefore the dynamic equation for the energy spectrum function $E(K)$ is

$$\left(\frac{\partial}{\partial t} + 2\nu K^2 \right) E(K,t) = T(K,t),$$

where $T(K,t)$ is related to $S_{ij}^{(2)}$ by $T(K,t) = \tfrac{1}{2}S_{ll}^{(2)}(K,t)$.

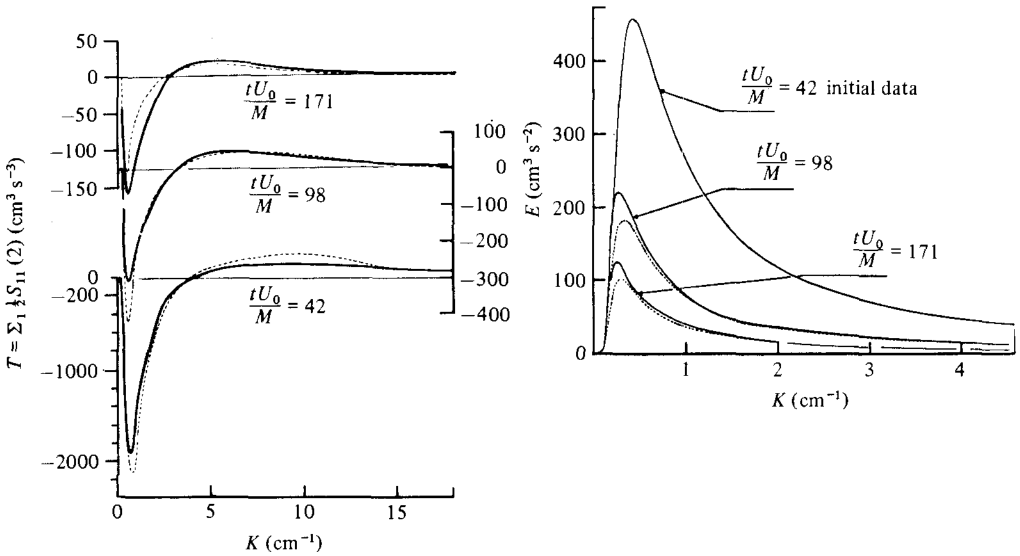


FIGURE 1. Downstream evolution of three-dimensional energy and transfer spectra. ——, data from Comte-Bellot & Corrsin (1971), $M = 5{\cdot}08$ cm, $U_0 = 10$ m s^{-1}; ---, calculated with the EDQNM theory.

Under its simplified form, the previous model leads to the classical EDQNM theory for isotropic turbulent fields.

From Pouquet *et al.* (1975) the eddy damping rate η can be taken in the form

$$\eta(K,t) = \lambda\left[\int_0^K P^2E(P,t)\,dP\right]^{\frac{1}{2}},$$

despite the fact that the expression is not well justified at small K, where $\eta(K) \sim K^2$ appears more acceptable.

The decay of an isotropic turbulence is predicted and compared with experimental data of Comte-Bellot & Corrsin (1971). The selected value of the constant $\lambda(= 0{\cdot}360)$ has already been chosen by André & Lesieur (1977) referring to the test field model results in inertial range for large values of the turbulent Reynolds number. It is convenient for this case (figure 1) and seems suitable for various turbulent fields.

In order to determine the second function $a(K,t)$, comparisons are made for the case of a rapid distortion. Where a pure straining is applied to an isotropic turbulent field, the analytical solution of equation (2.3) can be compared with (3.4). The compatibility between the two expressions leads to the relation

$$a(K,t) = -\frac{3}{2}\frac{[e_{ln}(t)\,e_{ln}(t)]^{\frac{1}{2}}}{[H_{ln}(K,t)\,H_{ln}(K,t)]^{\frac{1}{2}}}, \tag{5.1}$$

with

$$e_{ij}(t) = \frac{C_{ij}(t)}{C_{ll}(t)} - \frac{\delta_{ij}}{3}.$$

The expression (5.1) for $a(K,t)$ is taken for any homogeneous fields; for large values of K, when $\mathbf{H}$ goes to zero, it is supplemented by the restriction

$$|a(K,t)| < a_0, \quad a_0 = 4{\cdot}5.$$

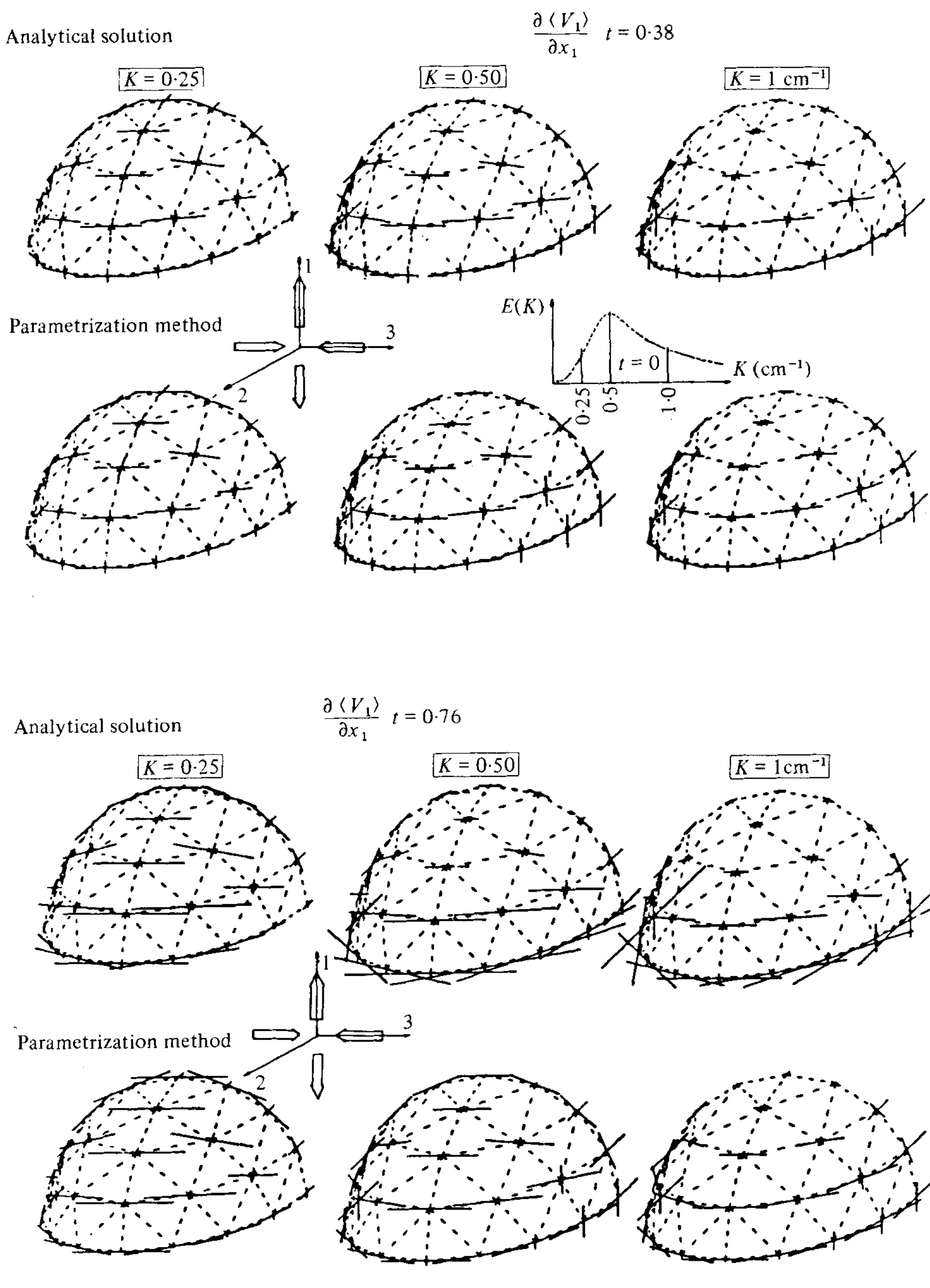
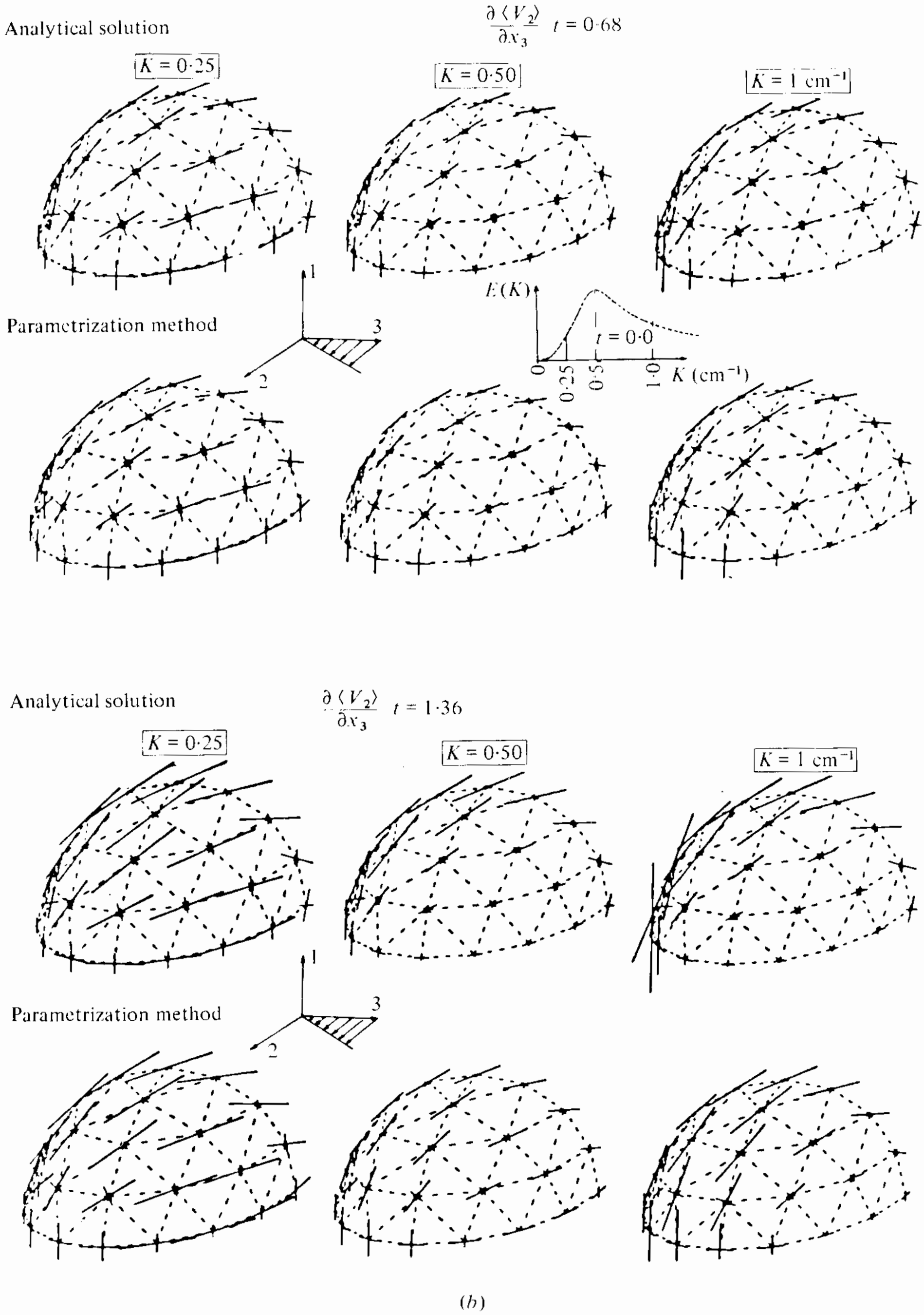


FIGURE 2. Comparison between analytical linear solution and parametrization method for the rapid distortion of initially isotropic turbulence (a) by plane irrotational strain and (b) by plane shear. Angular shape of second-order spectral tensors.



(b)

FIGURE 2 (b). For legend see p. 255.

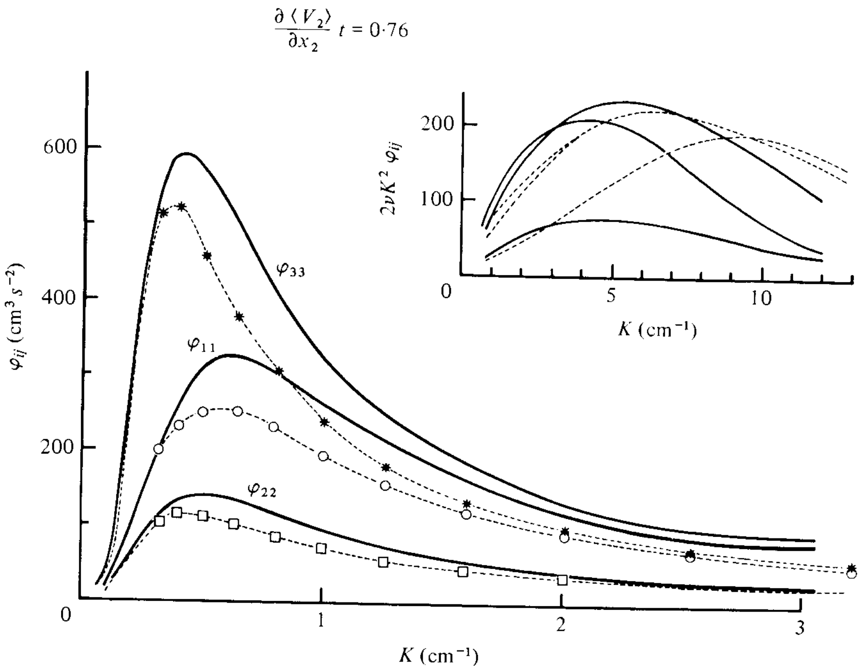


FIGURE 3. Plane irrotational strain applied to initially isotropic turbulence. Initial data from Comte-Bellot & Corrsin (1971). Spectrum functions $\varphi_{ij}(K)$ and $2\nu K^2 \varphi_{ij}(K)$. ——, linear solution; - - -, present model.

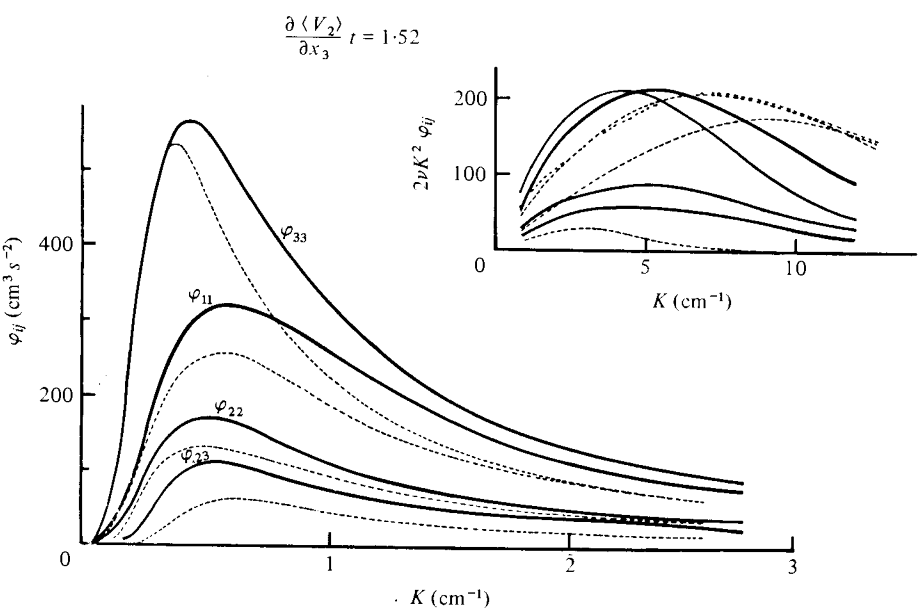


FIGURE 4. Plane shear applied to initially isotropic turbulence; initial data from Comte-Bellot & Corrsin (1971). Spectrum functions $\varphi_{ij}(K)$ and $2\nu K^2 \varphi_{ij}(K)$ in the principal axes of the pure distortion. ——, Linear solution; - - -, present model.

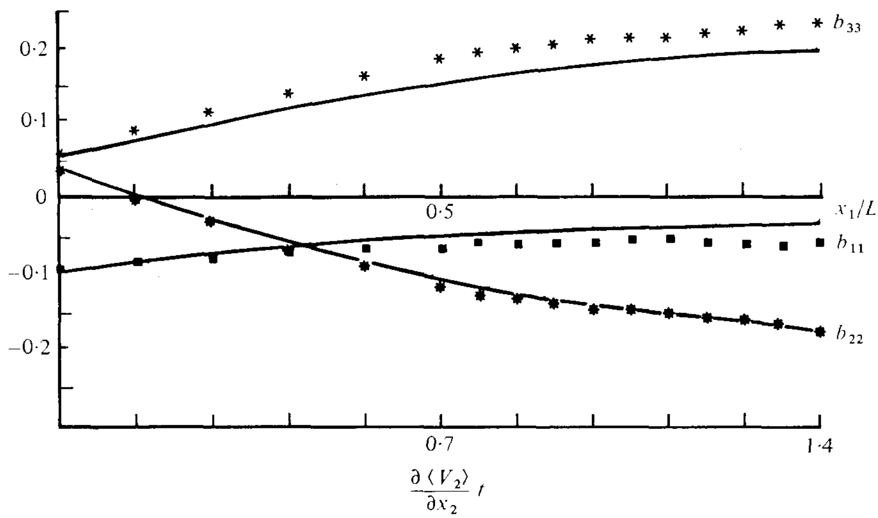


FIGURE 5. Plane irrotational strain; downstream evolution of the deviatoric tensor b_{ij}. $*$, ■, ✱, experimental data from Gence (1979); ——, present model.

Starting from the linear form of equation (2.3), analytical solutions are computed for two particular homogeneous flows subjected to a straining process; the mean values φ_{ij} are deduced; from the corresponding value of $\mathbf{H}$, it is possible to evaluate $\mathscr{T}_{ij}$. In figures 2 (*a*) and (*b*), comparisons are made between the angular shape of the analytical solution of (2.3) and that obtained by the parametrization method. For each angular position, in the plane normal to $\mathbf{K}$, the two principal axes to ϕ_{ij} are drawn and their lengths are proportional to eigenvalues.

6. Numerical results

The closed equation (3.1) is numerically solved. First, we examine the role played by turbulent structures as a function of their sizes. An initially isotropic turbulence suddenly subjected to either a pure strain or a shear flow is considered. Computations are carried out from the complete equation (3.1); similar computations are also performed when the nonlinear terms are dropped out. Spectral results are compared in figures 3 and 4. For small values of K, the linear effects ($S_{ij}^{(1)}$, $P_{ij}^{(1)}$) prevail, whereas the isotropy of the small structures is ensured by nonlinear effects.

Secondly, a comparison is made with experimental data given by Gence (1979). A quasi-isotropic turbulence is subjected to successive plane strains.

For lack of information, additional hypothesis are made in order to relate initial conditions on φ_{ij} to available measurements.

The energy spectrum $E(K, 0)$ is deduced from one-dimensional data $E_{ll}(K_1, 0)$:

$$E(K,0) = \tfrac{1}{2}K \left[\frac{d}{dK_1} E_{ll}(K_1, 0)\right]_{K_1 = K}.$$

The small anisotropy, at the entrance of the distorting duct, is taken into account by supposing the initial anisotropy to be generated by an hypothetic strain λ_{ij}^0 acting on an isotropic turbulence during a short time Δt^0. It can be shown that the kinetic

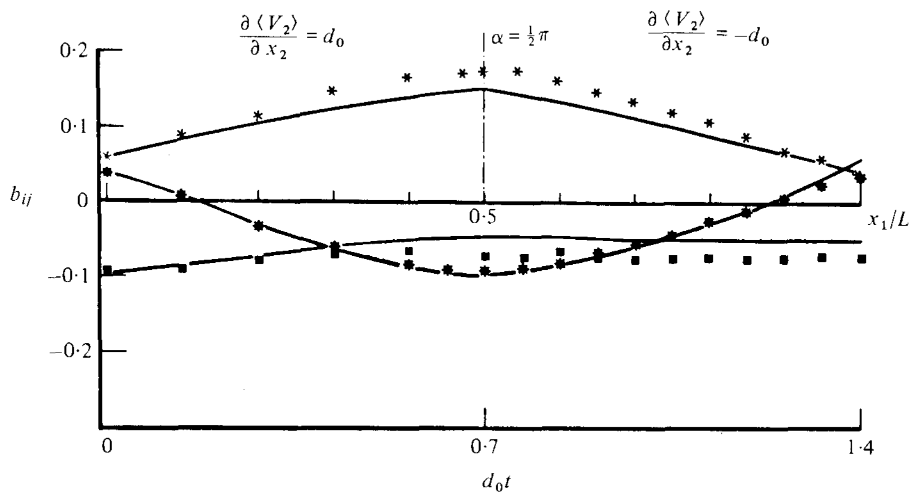


FIGURE 6. Application of two successive plane strains of opposite sign. Downstream of the deviatoric tensor b_{ij}. *, ■, ✱, experimental data from Gence (1979); ——, present model.

turbulent energy is not significantly altered by this hypothetic strain, whereas the small anisotropy is governed by

$$H_{ij}(K,0) = -\frac{1}{6}\left[1+\frac{1}{5}\left(\frac{K}{E}\frac{\partial E}{\partial K}\right)_{t=0}\right](\lambda^0_{ij}+\lambda^0_{ji})\,\Delta t^0,$$

$$b_{ij}(0) = \frac{\langle v_i v_j\rangle(0)}{\langle v_l v_l\rangle(0)} - \frac{\delta_{ij}}{3} = -\tfrac{2}{15}(\lambda^0_{ij}+\lambda^0_{ji})\,\Delta t^0,$$

from which we also have

$$H_{ij}(K,0) = \frac{1}{4}\left[5+\left(\frac{K}{E}\frac{\partial E}{\partial K}\right)_{t=0}\right] b_{ij}(0),$$

$$e_{ij}(0) = -\tfrac{15}{2} b_{ij}(0).$$

In a first stage the initially quasi-isotropic turbulence is subjected to a unique straining process. Figure 5 shows the deviatoric tensor b_{ij} as a function of the strain ratio.

In a second stage, the same isotropic field is successively subjected to two plane strains of opposite signs. The components of b_{ij} are presented in figure 6.

For the cases where the principal axes of the second strain have been rotated through an angle α with respect to the first one, comparisons are made in figure 7 (*a*) between the computed values of the second invariant b_{ij} and experimentsl data.

The model underestimates the level of anisotropy; the adaptativity of the mean flow to the sudden change of the strain axes is not instantaneous, so that some discrepancies should be observed. In the same conditions a rapid distortion theory is applied to equation (2.3); the results are shown in figure 7 (*b*).

When the mean velocity gradient is dropped, the anisotropic turbulence coming from the distorting duct is analysed in figure 8. In physical space, Rotta (1951) suggested an appropriate form for the nonlinear part of the rate-of-strain pressure correlation.

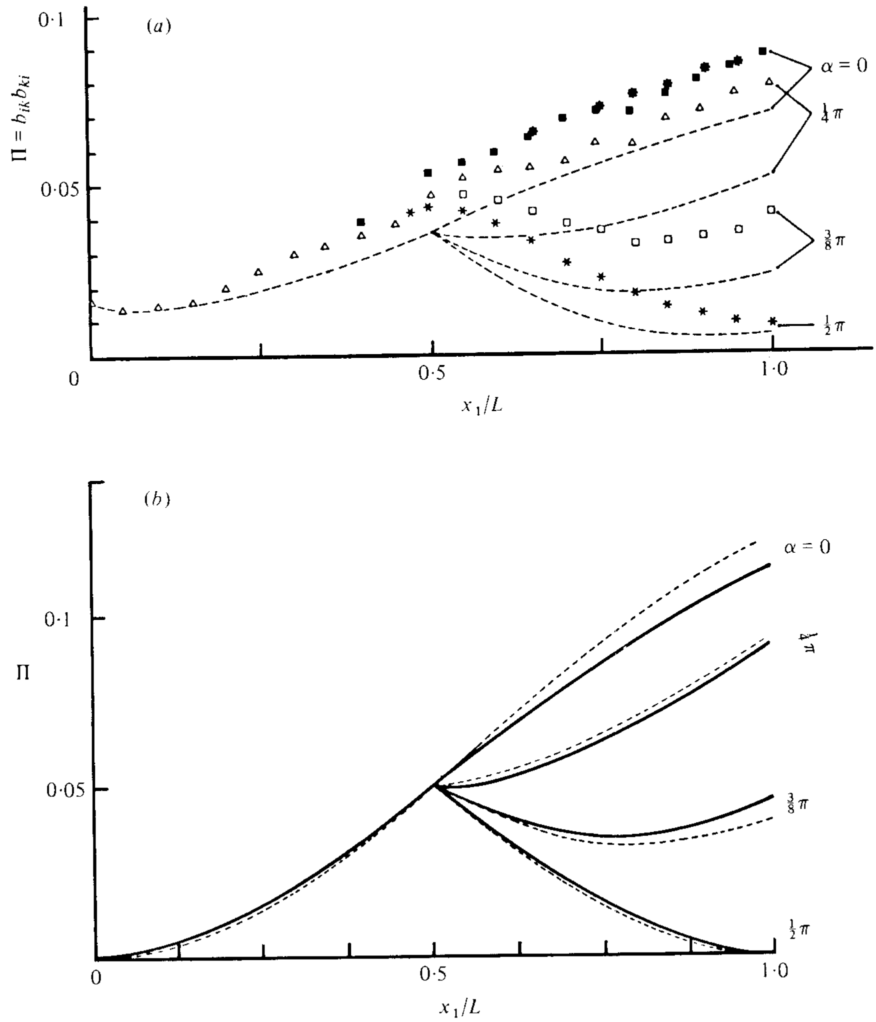


FIGURE 7. Application of two successive plane strains rotated through an angle α. (a) Downstream evolution of the invariant $\Pi = b_{ij}b_{ji}$. ✶, □, △, ■, ✱, experimental data from Gence (1979); ---, present model. (b) Rapid distortion. Evolution of the invariant $\Pi = b_{ij}b_{ji}$. ——, analytical solution; ---, present model.

This term is responsible for the intercomponent energy exchange and the following form is proposed:

$$\left\langle \frac{p}{\rho}\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)\right\rangle = -2C_1\bar{\epsilon}b_{ij};$$

$\bar{\epsilon}$ is the rate of turbulent energy dissipation and Lumley (1975) proposed for C_1

$$C_1 = 1{\cdot}5(1+7{\cdot}446b_{ij}b_{ji}). \tag{6.1}$$

With the spectral model, appropriate values of $C^{(ij)}$ are computed:

$$C^{(ij)} = -\int_0^\infty P_{ij}^{(2)}(K,t)\,dK/2\bar{\epsilon}\,b_{ij}.$$

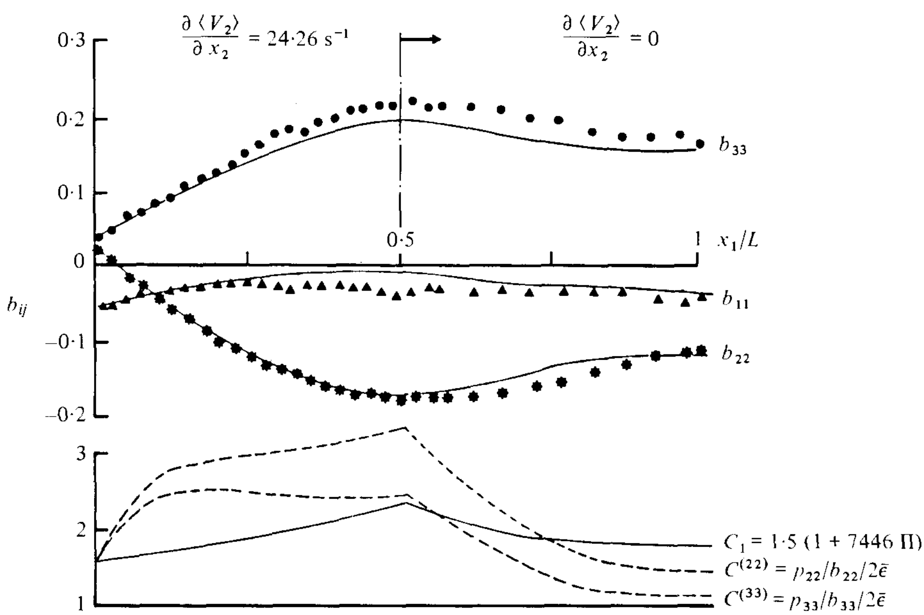


FIGURE 8. Return to isotropy of distorted turbulence. Downstream evolution of b_{ij} and corresponding evolution of the rates of decrease of anisotropy. ●, ▲, ✱, experimental data from Gence (1979); ——, present model.

These values are roughly consistent with those given by the approach of Lumley. In the distorting part of the duct, smaller values are deduced from (6.1) whereas larger ones are obtained where the mean velocity gradient is dropped out. In this last situation (return to isotropy), the results presented herein seem in agreement with those of Schumann & Herring (1976).

7. Conclusion

The method presented herein is an extension of the EDQNM theory to moderately anisotropic turbulent flows subjected to mean velocity gradients. In order to simplify the computing approach, a spectral function is introduced which includes the overall informations required to obtain final results depending on the modulus of wavenumber K only.

The action of the mean field is explicitly taken into account through the linear terms of the equation governing the double correlations. Where the equation for triple correlations is concerned, the role played by both viscous terms and the fourth-order cumulants are grouped into a same term by introducing an eddy damping coefficient. For convenience and also for lack of information, the term which takes explicitly into account the mean velocity gradient is also grouped with the two previous one. Accordingly, the anisotropy of the turbulence is mainly introduced through the term Ω^{QN}_{ijl} related to the double correlations. Such an assumption leads to an equation which is treated by means of a Markovian hypothesis.

A good agreement with typical data is found. Some improvements will be made possible when additional experimental data become available. Three-dimensional spectra are necessary to support this kind of theory in detail.

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